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Sedimentation of a two-dimensional drop towards a rigid horizontal plane

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We consider the drainage of fluid trapped beneath a two-dimensional drop that sediments towards a horizontal plane. The governing equation is closely related to that for capillary drainage (without gravity) of an annular film discussed in a companion paper (Lister *et al.*, *J. Fluid Mech.* vol. 552, 2006, p. 311). When drainage starts, dynamical structures rapidly appear that are usually called *dimples* in the context of sedimentation. Dimples are constant-pressure regions to which most of the fluid in the film is confined, which are analogous to the *collars* and *lobes* that appear in annular capillary drainage.

The process of drainage is controlled by a Bond number, B, that measures the relative importance of gravity and surface tension for the sedimenting drop. When B is sufficiently small, all the fluid ultimately drains from a single small dimple and the drop takes a static sessile equilibrium shape. The dimple-drainage process is the same as that of a lobe. When B is sufficiently large, several permanent dimples are formed under the drop, and these exhibit complex dynamics of collision and interaction analogous to that of collars and lobes. No static drop shape is reached, even for long times. For critical values of B, fluid may be permanently trapped in one or more stationary dimples (analogous to collars), and families of equilibrium drop shapes are found that depend upon the quantity of trapped fluid.

1. Introduction

When a drop of one fluid settles through another fluid onto a rigid horizontal plane it is well-known that some of the external fluid becomes trapped in a squeeze film between the drop and the plane (e.g. Jones & Wilson 1978; Yiantsios & Davis 1990; Ascoli, Dandy & Leal 1990). In the absence of van der Waals forces, the flow in the film is driven by a combination of hydrostatic and capillary pressure and is described by lubrication theory. The governing equation is closely related to that for capillary drainage of an annular film discussed in Lister *et al.* (2006, hereinafter I), and, as we demonstrate below, the main features of the evolution correspond.

If the film drains completely, then at long times, the drop adopts a static shape with a flat bottom, and this sessile shape is (uniquely) determined by the Bond number

$$B = \frac{\Delta\rho\, g b^2}{\sigma}, \tag{1.1}$$

where $\Delta\rho$ is the density excess of the drop and b is its undeformed radius; g is acceleration due to gravity and σ is surface tension. For $B \ll 1$ the sessile drop is near-spherical, and at long but finite times t the almost flat bottom takes the form of a small paraboloidal dimple of width $O(B^{1/2})$ and amplitude $O(t^{-1/4})$. The fluid trapped in the dimple escapes slowly through an annular neck of width $O(t^{-1/4})$ and height $O(t^{-1/2})$ (Jones & Wilson 1978) in the same way as a single stationary lobe (in the language of I) drains into adjacent stationary collars (Hammond 1983). For $B \gg 1$ the equilibrium drop shape is a low-lying, wide pancake, and the size of the flat bottom is sufficient to accommodate trapped dimples with shapes analogous to collars. The fluid trapped in collar-like dimples does not drain, just as for collars in I.

In this paper we present a brief investigation into the possibility of complex drainage behaviour underneath sedimenting drops at large Bond number. For simplicity, we consider only two-dimensional drops. We do not consider the complex deformation of large-Bond-number drops far from the wall (Ascoli *et al.* 1990; Stone 1994) but instead focus on the late-stage behaviour when the drop has already settled into a quasi-static shape.

2. Governing equation

For a quasi-static drop shape, the modified pressure just outside the drop is given by

$$p = \frac{\sigma P_d}{b} - \Delta\rho g h - \frac{\sigma h_{xx}}{\left(1+h_x^2\right)^{3/2}}, \tag{2.1}$$

where $\sigma P_d/b$ is the pressure in the drop and $h(x,t)$ is the height of the drop surface above the plane. The height is two-valued: the upper surface of the drop is given by the equilibrium condition $p=0$; below the drop a pressure gradient p_x drives flow in a squeeze film that is described by lubrication theory. We scale lengths with b and times with $\mu b/\sigma$, where μ is the external fluid viscosity, to obtain the film evolution equation

$$h_t + \left(\frac{h^3}{3}\left(Bh + \frac{h_{xx}}{\left(1+h_x^2\right)^{3/2}}\right)_x\right)_x = 0. \tag{2.2}$$

We assume that the internal drop viscosity is not so high as to modify the film flow. At the edges of the film the height can be matched to the equilibrium upper surface shape since $p_x \to 0$ as h^3 increases rapidly. The term involving B corresponds to the destabilizing effect of gravity acting on the buoyant fluid trapped below the drop; the final term corresponds to the stabilizing effect of surface tension. This equation can also be used to describe the evolution of a thin film coating the underside of a ceiling (e.g. Yiantsios & Higgins 1989; Ehrhard 1994).

Equation (2.2) is closely related to the equation

$$h_t + \left(\tfrac{1}{3}h^3(h+h_{zz})_z\right)_z = 0 \tag{2.3}$$

studied in I for the evolution of the thickness $h(z,t)$ of an annular film, and can be reduced to it in the limit $h_x^2 \ll 1$ by a rescaling of x by a factor $B^{1/2}$. Conversely, the governing equation for drainage of an annular film can be extended by retaining the full curvature of an axisymmetric film (equation (2.1) in I) instead of the small-amplitude approximation $h + h_{zz}$. The retention of the $O(h_x^2)$ corrections to the curvature in (2.2), despite using lubrication theory, is a rational approximation in the limit where the dynamics are controlled by slow processes in narrow necks for which $h_x^2 \ll 1$ and lubrication theory holds; the intervening collars and lobes have quasi-static shapes that depend on the full curvature at large amplitudes (Gauglitz & Radke 1988).

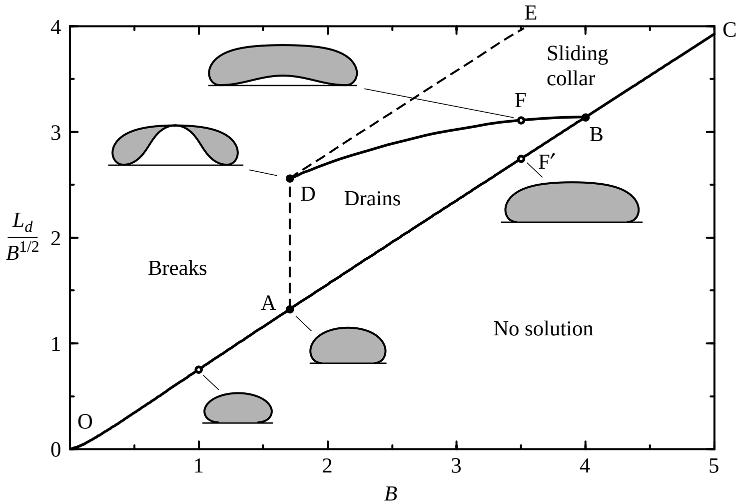

FIGURE 1. Regime diagram for drops of different lengths and cross-sectional areas. See text for description. $L_d/B^{1/2}$ as a function of B.

3. Equilibrium drop shapes

The shapes of sessile drops with no underlying trapped fluid are well-known (see e.g. Hodges, Jensen & Rallison 2004). In particular, for $B \gg 1$ the drop makes contact with the plane ($h = 0$) over a length $L_d \approx (\pi/2)B^{1/2}$, with exponentially small corrections, and the static balance gives $P_d = \pi B/L_d$. Other equilibria are possible only if fluid is permanently trapped under the drop and, as shown in I, this can arise for 'collar'-like dimples.

'Collar'-like static dimple solutions to (2.2) are given by

$$Bh + \frac{h_{xx}}{\left(1 + h_x^2\right)^{3/2}} = P_d - p_c \tag{3.1}$$

with $h_x = 0$ at the ends $x = x_0 \pm \ell/2$ of the dimple, where p_c is the constant pressure in the 'collar'. The term 'collar' in this context is somewhat of an abuse of language in that it suggests an annular structure. For that reason we enclose it in quotation marks. Nevertheless, it is essential to distinguish between 'collar'-like and 'lobe'-like dimples. The shorter 'lobes' also satisfy (3.1) but have $h_x \neq 0$ at their ends: a 'lobe' makes a finite contact angle with the plane. In contrast to the solutions based on a linearized curvature in I, 'collar' lengths and pressures here depend on the volume of fluid that they contain. A 'collar' of infinitesimal amplitude A has the profile $h = A[1 + \cos(B^{1/2}x)]$ (for $x_0 = 0$) and so $\ell = 2\pi B^{-1/2}$. The length $\ell(A)$ decreases monotonically as the amplitude increases and the shape deviates from the small-amplitude form.

In figure 1 we show the calculated regime diagram for quasi-static shapes of a drop with contact length L_d as a function of B: the line OC describes the branch of sessile-drop solutions with no trapped fluid; BD describes a branch of static equilibrium solutions which have a single underlying 'collar' of trapped fluid of exactly the same length as the drop ($\ell = L_d$); at D the 'collar' has the same height as the drop, so that to the left of D the drop must break into two or more sub-drops. Other equilibrium branches of solutions (not shown in figure 1) containing more than one 'collar'

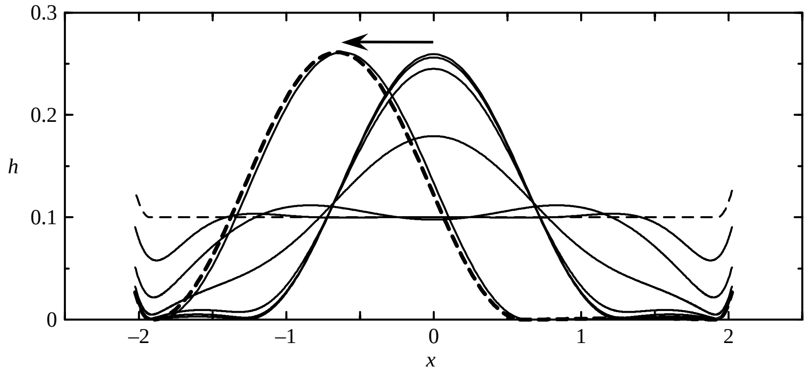

FIGURE 2. Sedimentation with $B = 6$ and $h_0 = 0.1$ showing successive profiles at $t = 0$ (short-dashed), $t = 10^{-2.5}, 10^{-1.5}, \ldots, 10^{3.5}$ (solid) and $t = 10^{4.5}$ (dashed). The initial conditions (short-dashed) are approximately symmetric, and for $t < 10^3$ the profile remains approximately symmetric with a single central 'collar' and equal 'lobes' on either side. The central position of the trapped 'collar' is unstable, and the 'collar' slides to the left when $t \approx 10^3$ breaking the symmetry and leaving a single 'lobe' on the right.

bifurcate from the line OC when $B \approx 8, 12, \ldots$. The drop shapes depend upon the relative sizes of the 'collars' involved.

In the region ABD the drop is shorter than the 'collar' of appropriate area, and hence the dimple instead is 'lobe'-like and drains by the Jones & Wilson mechanism until the relevant point on AB is reached; thus small perturbations to drops at D and the representative point F result in drainage to A and F′, respectively. In the region EDBC the drop is longer and higher than the relevant 'collar', which thus has room to slide sideways beneath the drop; along DE the 'collar' has the same height as the centre of the drop.

4. Film drainage

To illustrate some of the dynamical possibilities raised by figure 1, we solved (2.2) numerically for the lower boundary $h(x)$ of the drop on a domain $-L \leqslant x \leqslant L$, where $L = L_d(B) + 0.25$. The initial conditions were that the drop shape should be that of the sessile drop for that value of B lifted by some constant height h_0 (typically 0.1) above the plane. This prescription bypasses the details of deformation far from the wall (Ascoli *et al.* 1990) in order to focus on the phenomena of long-term drainage; the results depend only weakly on the initial conditions. The boundary conditions were that the curvature and slope of the interface at $x = \pm L$ should continue to equal those of the sessile shape. This prescription approximates matching to the sessile top surface of a drop of appropriate area, but neglects the increase in L_d if there is a significant amount of trapped fluid. Computations are more difficult for this problem than those described in I in view of the large curvature variation at the edge of the film.

For $B \lesssim 4$ we find that the trapped fluid drains from a single 'lobe'-like dimple according to the similarity scalings

$$h(x, t) = t^{-1/2} f(\xi), \quad \xi = (x - L_d)t^{1/4}, \tag{4.1}$$

of Jones & Wilson (1978) and Hammond (1983), where f is a similarity function satisfying the constant-flux condition $f^3 f''' = \text{const.}$

For $B \gtrsim 4$, the trapped fluid forms at least one 'collar' with the remaining space taken up by 'lobes'. In figure 2 we show the evolution for $B = 6$ and $h_0 = 0.1$. For

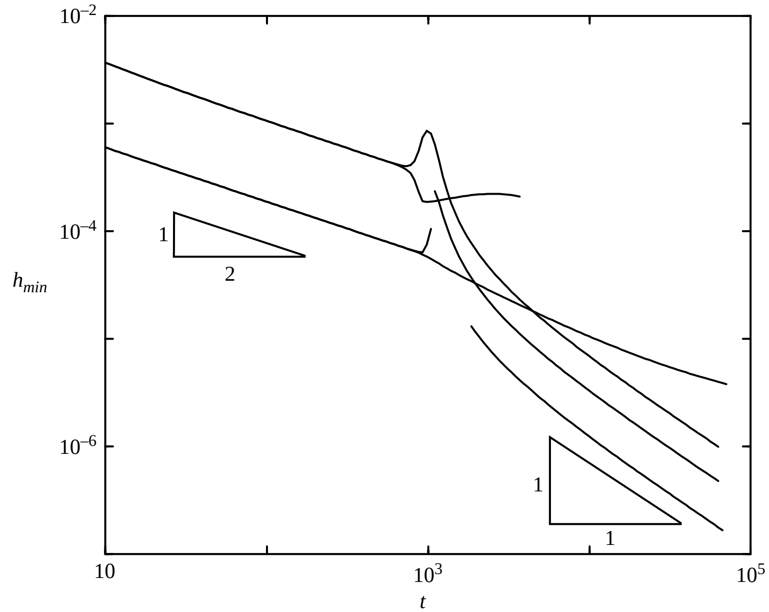

FIGURE 3. The heights of the various minima in the calculation with $B = 6$ shown in figure 2. There is a clear change in scaling from $t^{-1/2}$ to t^{-1} when the 'collar' slides from its central position to the left of the drop at $t \approx 10^3$.

early times a central 'collar' forms with 'lobes' on either side that drain both into the 'collar' and out from under the drop. At this stage the various minima decrease like $t^{-1/2}$ (figure 3), as in the similarity solution of Jones & Wilson (1978) and Hammond (1983). However, the central position for the 'collar' becomes unstable around $t = 10^3$ and, triggered by numerical noise, the 'collar' slides to the left until it meets the edge of the drop. The sliding motion occurs by the same mechanism and on the same timescales as those obtained in I and, in particular, the minima ahead of and behind the 'collar' then decrease like t^{-1}. We did not pursue the calculation long enough to observe 'peeling' at the trailing minimum and sliding to the other side (cf. § 7 of I), though it would presumably occur. This calculation and others support the expectation that the squeeze film under a sedimenting drop at large Bond number displays the same complex dynamics as the annular film on a long domain. Neither the difference in boundary conditions nor the use of the finite-amplitude curvature expression plays a significant role, other than in the definition of quasi-static shapes and in the critical length.

We also explored the evolution at the critical Bond number, where the trapped fluid forms a single 'collar' of length exactly equal to that of the drop. As shown in figure 1, the critical Bond number decreases from about 4 as both h_0 and the consequent amplitude of the trapped 'collar' increase. The final shapes correspond closely to those along BD in figure 1, but are not exactly the same owing to the approximations made to the boundary conditions at the edge of the drop. The evolution of these shapes is governed by the same similarity equations as for a stationary collar when $L = 2\pi$ in § 4 of I, with the necks described by

$$h(x,t) = t^{-1}F(\zeta), \quad \zeta = (x - \ell/2)t^{1/2}, \tag{4.2}$$

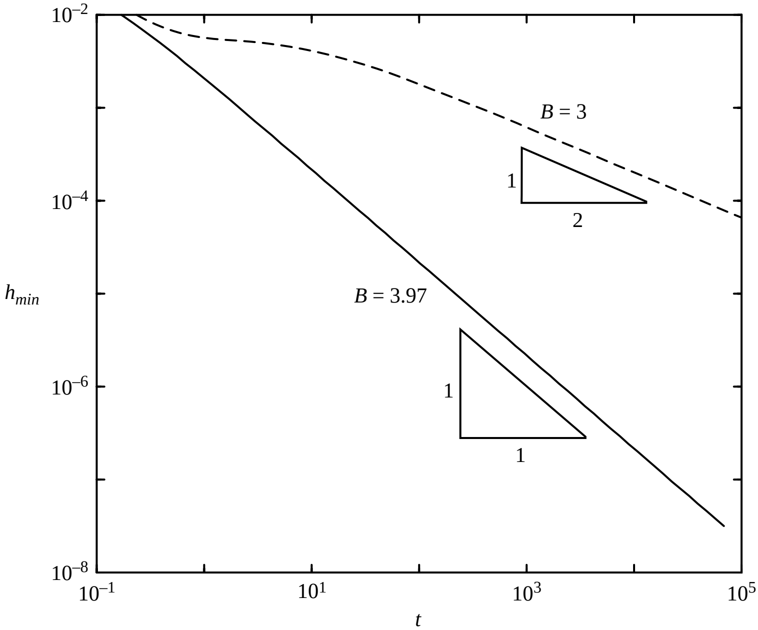

FIGURE 4. The heights of the minima on either side of the central dimple for $B = 3.97$ (solid) and $B = 3$ (dashed) with $h_0 = 0.1$. For $B = 3.97$ and $h_0 = 0.1$ the central dimple is a 'collar' and the scaling $h_{min} \propto t^{-1}$ is in accordance with the similarity form (4.2); for $B = 3$ the central dimple is a shorter 'lobe' and the scaling $h_{min} \propto t^{-1/2}$ is in accordance with the similarity solution of Jones & Wilson (1978).

where

$$-F + \tfrac{1}{2}\zeta F' + \tfrac{1}{3}(F^3 F''')' = 0, \tag{4.3}$$

$$F = \tfrac{1}{2}A_{\pm}\zeta^2 + 0.\zeta + O(1) \quad \text{as} \quad \zeta \to \pm\infty \tag{4.4}$$

and ℓ is the length of the trapped 'collar'. An example of the similarity scaling $h \sim t^{-1}$ is shown in figure 4 for $B = 3.97$, which is the critical value for $h_0 = 0.1$. The height profiles in the neck at different times can also be collapsed using the similarity variables and shown to give good agreement with the relevant solution of (4.3) and (4.4), as in figure 10 of I.

In equation (4.4) the value of A_+ is the curvature at the edge of the sessile top surface; the value of A_- depends on the amplitude of the trapped collar and can be varied by changing h_0. The rescaling $F \to \lambda^4 F$, $\zeta \to \lambda^3 \zeta$, where λ is a constant, leaves (4.3) and (4.4) invariant and rescales $A_{\pm} \to \lambda^{-2} A_{\pm}$. Thus the similarity solutions for the critical Bond number can be reduced to a one-parameter family, defined only by the curvature ratio A_-/A_+.

5. Conclusions

The drainage of the film beneath a sedimenting drop of sufficient size displays a rich dynamics on several different timescales analogous to those of an annular film explored in I. In addition to the familiar sessile drop shape where the drop makes contact with the plane along its lower surface, a class of static drop shapes for which fluid is permanently trapped beneath the drop may be found at critical Bond numbers. These unfamiliar equilibria are unstable owing to the increase in the length of a 'collar' with a decrease in its amplitude.

At larger Bond numbers some fluid is permanently trapped beneath the drop and the film evolves on progressively longer timescales as 'collar'-like dimples slide and collide underneath the drop. No static drop shape is attained for finite times. The mechanism for permanent trapping is the rapid decrease of the minimum thickness at the leading edge of any 'collar' approaching the sessile edge of the drop, just as for the non-coalescence of one collar approaching another analysed in I.

The phenomena described here arise for two-dimensional drops. We anticipate that similar dynamics also arise in the three-dimensional case.

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